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A DISCRETE NON-CLASSICAL OPERATIONAL CALCULUS MODEL WITH THE HORADAM DIFFERENCE

ABSTRACT

In this paper, there has been constructed such a model of the non-classical Bittner operational calculus, in which the derivative $S$ related to Horadam sequences is understood as a difference operation $S\{x(k)\} := \{x(k + 2) - a x(k + 1) - b x(k)\}$.

Key words: operational calculus, derivative, integrals, limit conditions, Horadam difference.

INTRODUCTION

For any functions $\{f(t)\} \in C^0((\alpha, \beta), \mathbb{R})$, $\{x(t)\} \in C^1((\alpha, \beta), \mathbb{R})$ as well as for every $t_0 \in (\alpha, \beta) \subset \mathbb{R}$ and $t \in (\alpha, \beta)$ the fundamental theorems of the integral calculus apply [1]:

$$\frac{d}{dt} \int_{t_0}^{t} f(\tau) \, d\tau = f(t), \quad \int_{t_0}^{t} x'(\tau) \, d\tau = x(t) - x(t_0).$$

Using linear operations

$$S\{x(t)\} := \{x'(t)\}, \quad T_{t_0}\{f(t)\} := \left\{\int_{t_0}^{t} f(\tau) \, d\tau \right\}, \quad s_{t_0}\{x(t)\} := \{x(t_0)\},$$

we can present the above theorems as follows:

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Apart from the model (1) with the classical ordinary derivative \( S = d/dt \), there exist other continuous and discrete models in which, for appropriately determined operations \( S, T_q, s_q \) properties (2) hold. These models constitute particular cases (representations) of the so-called non-classical Bittner operational calculus [2–5].

Broadly speaking, the Bittner operational calculus is a system

\[
CO(L^0, L^1, S, T_q, s_q, Q),
\]

in which \( L^0 \) and \( L^1 \) are linear spaces (over the same scalar field \( F \)) such that \( I^1 \subseteq I^0 \). The linear operation \( S : L^1 \rightarrow L^0 \) (denoted as \( S \in \mathcal{L}(L^1, L^0) \)), called the (abstract) derivative, is a surjection. Moreover, \( Q \) is a set of indices \( q \) for the operations \( T_q \in \mathcal{L}(L^0, L^1) \) and \( s_q \in \mathcal{L}(L^1, L^1) \) such that \( S T_q f = f, f \in L^0 \) and \( s_q x = x - T_q S x, x \in L^1 \). These operations are called integrals and limit conditions, respectively. The kernel of \( S \), i.e. \( \ker S \) is a set of elements understood as constants for the derivative \( S \). The limit conditions \( s_q, q \in Q \) are projections of \( L^1 \) on the subspace \( \ker S \).

Beside the continuous model (2), we frequently use a classical discrete model with the derivative \( S \) understood as the forward difference \( \Delta \).

Let \( \mathbb{N}_0 \) and \( \mathbb{C} \) mean the set of non-negative integers and the set of complexes, respectively. Moreover, let \( L^0 := C(\mathbb{N}_0, \mathbb{C}) \) be a linear space of complex sequences \( x = \{x(k)\}_{k \in \mathbb{N}_0} \) with usual sequences addition and sequences multiplication by complexes. In [2, 3, 5] Bittner considered a model with the derivative

\[
S x \equiv \Delta x := \{x(k + 1) - x(k)\}
\]

and its corresponding integral

\[
T_0 x := \left\{ \begin{array}{ll}
0 & \text{for } k = 0 \\
\sum_{i=0}^{k-1} x(i) & \text{for } k > 0
\end{array} \right., \quad k \in \mathbb{N}_0
\]

and limit condition

\[
s_0 x := \{x(0)\},
\]

where \( x = \{x(k)\} \in L^1 = L^0 \).

---

1. \( \{f(t)\} \) stands for the symbol of the function \( f \), i.e. \( f = \{f(t)\} \) whereas \( f(t) \) denotes the value of the function \( f(t) \) at point \( t \). This notation is derived from J. Mikusiński [15].

2. \( CO \) stands for the French ‘calcul opératoire’ (operational calculus).
Later, in [6] there appeared a model with the forward difference \( S \equiv \Lambda \), integrals

\[
T_{k_0} x := \begin{cases} 
-k_{0}^{-1} \sum_{i=k}^{x(i)} & \text{for } k < k_0 \\
0 & \text{for } k = k_0 \\
-\sum_{i=k_0}^{x(i)} & \text{for } k > k_0 
\end{cases}
\]

and limit conditions

\[ s_{k_0} x := \{x(k_0)\}, \]

where \( k_0 \equiv q \in Q := \mathbb{N}_0 \).

Notice that the integrals \( T_{k_0} \) can be shown as follows

\[
T_{k_0} x = \left( \sum_{i=0}^{k-1} x(i) - \sum_{i=0}^{k_0-1} x(i) \right)^3.
\]

In this paper, we shall discuss other discrete models of the Bittner operational calculus related to the operation

\[
S \{x(k)\} := \{x(k + 2) - a x(k + 1) - b x(k)\}. \tag{4}
\]

where \( a, b \in \mathbb{C} \) and \( b \neq 0 \).

We will consider two cases:

\[
D := a^2 + 4b \neq 0 \quad \text{and} \quad D = 0.
\]

In literature (e.g. [7, 12, 14, 16]), each element \( c \) belonging to the kernel of the operation (4) is called a Horadam sequence [8, 9].

In particular, the Horadam sequence \( c = \{c(k)\} \in \text{Ker } S \), i.e. a solution of the equation

\[
c(k + 2) = a c(k + 1) + b c(k), \quad k \in \mathbb{N}_0, \tag{5}
\]

can be [10]:

- the Fibonacci sequence \( \{\mathcal{F}(k)\}\) (for \( a = b = 1, \mathcal{F}(0) = 0, \mathcal{F}(1) = 1 \))

\[
\{\mathcal{F}(k)\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots\};
\]

\[\text{Given the definition of } T_{k_0} \text{, we assume that } \sum_{i=0}^{k} x(i) := 0.\]
Another interesting example is also an anti-forward Fibonacci sequence \( \{ f(k) \} \) for which \( a = -1, b = 1, f(0) = 0, f(1) = 1 \). Then, we have
\[
\Delta f(k) = f(k + 2) = -[f(k + 1) - f(k)]
\]
from which we obtain
\[
\{ f(k) \} = \{ 0, 1, -1, 2, -3, 5, -8, 13, -21, 34, -55, 89, -144, 233, \ldots \}.
\]
We also have
\[
f(k) = (-1)^{k+1} f(k), \quad k \in \mathbb{N}_0.
\]  


**A MODEL WITH THE HORIZADAM DIFFERENCE, WHEN \( D \neq 0 \)**

In what follows, we shall call the operation (4) a Horadam derivative or difference.

Let
\[
\Phi_{a,b} := \frac{a + \sqrt{D}}{2}, \quad \Phi_{a,b} := \frac{a - \sqrt{D}}{2}, \quad D \neq 0.
\]
Then, we have
\[ \Phi_{a,b}^2 - a \varphi_{a,b} - b = 0, \quad \varphi_{a,b}^2 - a \varphi_{a,b} - b = 0 \]
and
\[ \Phi_{a,b} + \varphi_{a,b} = a, \quad \Phi_{a,b} - \varphi_{a,b} = \sqrt{D}, \quad \Phi_{a,b} \varphi_{a,b} = -b. \]

We will prove the following

**Theorem 1.** The system (3), where \( x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C}), k_0 \equiv q \in Q := \mathbb{N}_0 \) and
\[ S x := \{x(k + 2) - a x(k + 1) - b x(k), \quad (7) \]
\[ T_{k_0} x := \left\{ \frac{1}{\sqrt{D}} \left[ \sum_{i=k_0}^{k} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i}) x(i) \right] - \sum_{i=k_0}^{k-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i) \right\}, \quad (8) \]
\[ s_{k_0} x := \left\{ \frac{1}{\sqrt{D}} \left[ b (\Phi_{a,b}^{k_0-k} - \varphi_{a,b}^{k_0-k}) x(k_0) + (\Phi_{a,b}^{k_0} - \varphi_{a,b}^{k_0}) x(k_0 + 1) \right] \right\}, \quad (9) \]
forms a discrete model of the Bittner operational calculus with the Horadam difference (7), when \( D \neq 0 \).

**Proof.** It is obvious that (7) – (9) are linear operations. It is also easy to verify that \( T_{k_0} \) can be presented in the form of
\[ T_{k_0} x := \left\{ \frac{1}{\sqrt{D}} \left[ \sum_{i=0}^{k} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i}) x(i) - \sum_{i=0}^{k-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i) \right] \right\}, \]
where \( \sum_{i=0}^{j} x(i) := 0 \) for \( j = -2, -1 \).

Let \( \{y(k)\} := T_{k_0} \{x(k)\} \). Hence
\[ S \{y(k)\} = \{y(k + 2) - a y(k + 1) - b y(k)\} \]
\[ = \frac{1}{\sqrt{D}} \left[ \sum_{i=0}^{k} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i}) x(i) - \sum_{i=0}^{k-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i) \right] \]
\[ - a \left[ \sum_{i=0}^{k} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i}) x(i) - \sum_{i=0}^{k-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i) \right] \]
\[ - b \left[ \sum_{i=0}^{k} (\Phi_{a,b}^{k-i} - \varphi_{a,b}^{k-i}) x(i) - \sum_{i=0}^{k-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i}) x(i) \right] \]
\[
\begin{align*}
&= \left\{ \frac{1}{\sqrt{D}} \left( (\Phi_{a,b} - \varphi_{a,b})x(k) + (\Phi_{a,b}^2 - \varphi_{a,b}^2 - a(\Phi_{a,b} - \varphi_{a,b}))x(k - 1) \right) \\
&+ \sum_{i=0}^{k-2} \left[ (\Phi_{a,b}^2 - a \Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\varphi_{a,b}^2 - a \varphi_{a,b} - b)\varphi_{a,b}^{k-1-i} \right]x(i) \\
&- \sum_{i=0}^{k_0-1} \left[ (\Phi_{a,b}^2 - a \Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\varphi_{a,b}^2 - a \varphi_{a,b} - b)\varphi_{a,b}^{k-1-i} \right]x(i) \right\} = \{x(k)\},
\end{align*}
\]
so \( ST_{k_0}\{x(k)\} = \{x(k)\} \) holds.

Let \( \{f(k)\} := S\{x(k)\} = \{x(k + 2) - a \cdot x(k + 1) - b \cdot x(k)\} \), then

\[
T_{k_0}S\{x(k)\} = T_{k_0}\{f(k)\}
\]

\[
= \left\{ \frac{1}{\sqrt{D}} \left[ \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})f(i) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})f(i) \right] \right\}
\]

\[
= \left\{ \frac{1}{\sqrt{D}} \left[ \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i + 2) - \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i + 2) \right. \\
- a \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i + 1) + a \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i + 1) \\
- b \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) + b \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) \right\}
\]

\[
= \left\{ \frac{1}{\sqrt{D}} \left[ \sum_{i=2}^{k} (\Phi_{a,b}^{k+1-i} - \varphi_{a,b}^{k+1-i})x(i) - \sum_{i=2}^{k+1} (\Phi_{a,b}^{k+1-i} - \varphi_{a,b}^{k+1-i})x(i) \\
- a \sum_{i=1}^{k-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) + a \sum_{i=1}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) \\
- b \sum_{i=0}^{k-2} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) + b \sum_{i=0}^{k_0-1} (\Phi_{a,b}^{k-1-i} - \varphi_{a,b}^{k-1-i})x(i) \right] \right\}
\]

\[
= \left\{ \frac{1}{\sqrt{D}} \left( (\Phi_{a,b} - \varphi_{a,b})x(k) + (\Phi_{a,b}^2 - \varphi_{a,b}^2 - a(\Phi_{a,b} - \varphi_{a,b}))x(k - 1) \\
+ \sum_{i=2}^{k-2} \left[ (\Phi_{a,b}^2 - a \Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\varphi_{a,b}^2 - a \varphi_{a,b} - b)\varphi_{a,b}^{k-1-i} \right]x(i) \\
- (\Phi_{a,b}^{k-k_0} - \varphi_{a,b}^{k-k_0})x(k_0 + 1) \right\}
\]
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\[ - \sum_{i=2}^{k_0} \left[ (\Phi_{a,b}^2 - a \Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} - (\Phi_{a,b}^2 - a \Phi_{a,b} - b)\Phi_{a,b}^{k-1-i} \right] x(i) \]
\[ = \left\{ \frac{1}{\sqrt{D}} \left[ b(\Phi_{a,b}^{k-1-k_0} - \Phi_{a,b}^{k-1-k_0})x(k_0) + (\Phi_{a,b}^{k-k_0} - \Phi_{a,b}^{k-k_0})x(k_0 + 1) \right] \right\} \]

Therefore, \( T_{k_0} S \{x(k)\} = \{x(k)\} - s_{k_0} \{x(k)\} \) also holds, which completes the proof. \( \square \)

**Example 1.** It is not difficult to check (see Th. 3[5]) that an abstract differential equation

\[ Sx = f, \quad f \in L^0, \, x \in L^1 \]

with the limit condition

\[ s_0 x = c_{0,q}, \quad c_{0,q} \in \text{Ker} S \]

has exactly one solution

\[ x = c_{0,q} + T_{a,f}. \quad \text{(10)} \]

A. In particular,

\[ x(k) = \frac{1}{\sqrt{5}} \left[ \left( \frac{-1 + \sqrt{5}}{2} \right)^k - \left( \frac{-1 - \sqrt{5}}{2} \right)^k \right], \quad k \in \mathbb{N}_0 \]

is the solution of the homogeneous difference equation

\[ x(k + 2) + x(k + 1) - x(k) = 0, \quad k \in \mathbb{N}_0 \]

with initial conditions

\[ x(0) = 0, \, x(1) = 1, \]

which results from the limit condition form (9) for \( a = -1, \, b = 1 \) and \( k_0 = 0 \)

Hence we have

\[ x(k) = (-1)^{k+1} \cdot \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^k - \left( \frac{1 - \sqrt{5}}{2} \right)^k \right], \quad k \in \mathbb{N}_0. \]

It is a form (6) of the anti-forward Fibonacci sequence \( \{f(k)\} \) general term, where

\[ f(k) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^k - \left( \frac{1 - \sqrt{5}}{2} \right)^k \right], \quad k \in \mathbb{N}_0 \]

is a well-known Binet formula of the Fibonacci sequence \( \{F(k)\} \) general term.

B. For the Cauchy problem

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we have $a = 1, b = 2$ and for $k_0 = 0$, on the basis of (10), we obtain
\[
\mathcal{T}(k) = \frac{1}{3} \left[ 2^k - (-1)^k + 3 \sum_{i=0}^{k-2} (2^{k-1-i} - (-1)^{k-1-i}) \right] = \frac{1}{6} (2^{k+3} + (-1)^k - 9), \quad k \in \mathbb{N}_0.
\]
Similarly, if
\[
\mathcal{T}(k + 2) - \mathcal{T}(k) = 2^{k+2}, \quad k \in \mathbb{N}_0
\]
\[
\mathcal{T}(0) = 0, \mathcal{T}(1) = 1,
\]
then $a = 0, b = 1$ and for $k_0 = 0$, we get
\[
\mathcal{T}(k) = \frac{1}{2} \left[ 1 - (-1)^k + \sum_{i=0}^{k-2} (1 - (-1)^{k-1-i}) \cdot 2^{i+2} \right] = \frac{1}{6} (2^{k+3} + (-1)^k - 9), \quad k \in \mathbb{N}_0.
\]

The sequence $\{\mathcal{T}(k)\}$, defined with the use of Jacobsthal numbers as
\[
\mathcal{T}(k) := \begin{cases} 
  f(0) & \text{for } k = 0 \\
  f(1) & \text{for } k = 1 \\
  \sum_{i=0}^{k+1} f(i) & \text{for } k \in \mathbb{N}_0 \setminus \{0, 1\}
\end{cases},
\]
was introduced in [11], where Horadam gave a number of its properties, including (11) and (12).

A MODEL WITH THE HORADAM DIFFERENCE, WHEN $J = 0$

If the Horadam difference (4) takes a particular form of
\[
S\{x(k)\} := \left\{ x(k + 2) - a x(k + 1) + \frac{1}{4} a^2 x(k) \right\},
\]
then $J = 0$. For this case, we will prove.

**Theorem 2.** The system (3), where $x = \{x(k)\} \in L^0 = L^1 := C(\mathbb{N}_0, \mathbb{C})$, $k_0 \equiv a \in Q := \mathbb{N}_0$ and
\[
S\{x(k)\} := \left\{ x(k + 2) - a x(k + 1) + \frac{1}{4} a^2 x(k) \right\},
\]

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\[ T_{k_0} x := \left\{ \left( \frac{a}{2} \right)^{k-2} \left[ \sum_{i=0}^{k-2} \left( \frac{a}{2} \right)^i (k - 1 - i) x(i) - \sum_{i=0}^{k_0-1} \left( \frac{a}{2} \right)^i (k - 1 - i) x(i) \right] \right\}, \tag{14} \]

\[ s_{k_0} x := \left\{ -\left( \frac{a}{2} \right)^{k-k_0} \left[ (k - 1 - k_0) x(k_0) - \frac{2}{a} (k - k_0) x(k_0 + 1) \right] \right\}, \tag{15} \]

forms a discrete model of the Bittner operational calculus with the Horadam difference (13).

**Proof.** Similarly as before, the operations (13)–(15) are linear. Moreover, if \( \{y(k)\} := T_{k_0} \{x(k)\} \), then

\[
S \{y(k)\} = \left\{ y(k + 2) - a y(k + 1) + \frac{1}{4} a^2 y(k) \right\} \\
= \left\{ \sum_{i=0}^{k} \left( \frac{a}{2} \right)^{k-i} (k + 1 - i) x(i) - \sum_{i=0}^{k_0-1} \left( \frac{a}{2} \right)^{k-i} (k + 1 - i) x(i) \right\} \\
- 2 \sum_{i=0}^{k_1-1} \left( \frac{a}{2} \right)^{k-i} (k - i) x(i) + 2 \sum_{i=0}^{k_0-1} \left( \frac{a}{2} \right)^{k-i} (k - i) x(i) \\
+ \sum_{i=0}^{k_0-1} \left( \frac{a}{2} \right)^{k-i} (k - 1 - i) x(i) - \sum_{i=0}^{k_0-1} \left( \frac{a}{2} \right)^{k-i} (k - 1 - i) x(i) \right\} \\
= \left\{ x(k) + \sum_{i=0}^{k_0-1} \left( \frac{a}{2} \right)^{k-1-i} (k + 1 - i) x(i) - 2 (k - i) + (k - 1 - i) \right\} x(i) \\
- \sum_{i=0}^{k_0-1} \left( \frac{a}{2} \right)^{k-i} (k + 1 - i) x(i) + 2 (k - i) + (k - 1 - i) \right\} x(i) \right\} = \{x(k)\}. \\

If, in turn, \( \{f(k)\} := S \{x(k)\} = \{x(k + 2) - a x(k + 1) + \frac{1}{4} a^2 x(k)\} \), then

\[ T_{k_0} S \{x(k)\} = T_{k_0} \{f(k)\} \]

\[ = \left\{ \left( \frac{a}{2} \right)^{k-2} \left[ \sum_{i=0}^{k_0-1} \left( \frac{a}{2} \right)^i (k - 1 - i) f(i) - \sum_{i=0}^{k_0-1} \left( \frac{a}{2} \right)^i (k - 1 - i) f(i) \right] \right\} \]
\[
\left\{ \left( \frac{a}{2} \right)^{k-2} \sum_{i=0}^{k-2} \left( \frac{a}{2} \right)^i (k - 1 - i) x(i + 2) - \sum_{i=0}^{k_0-1} \left( \frac{2}{a} \right)^i (k - 1 - i) x(i + 2) \right\} \\
- a \sum_{i=0}^{k-2} \left( \frac{a}{2} \right)^i (k - 1 - i) x(i + 1) + a \sum_{i=0}^{k_0-1} \left( \frac{2}{a} \right)^i (k - 1 - i) x(i + 1) \\
+ \left\{ \left( \frac{a}{2} \right)^{k-2} \sum_{i=0}^{k-2} \left( \frac{a}{2} \right)^i (k - 1 - i) x(i) - \left( \frac{a}{2} \right)^{k_0-1} \sum_{i=0}^{k_0-1} \left( \frac{2}{a} \right)^i (k - 1 - i) x(i) \right\} \\
= \left\{ \sum_{i=2}^{k} \left( \frac{a}{2} \right)^{k-i} (k + 1 - i) x(i) - \sum_{i=2}^{k_0+1} \left( \frac{a}{2} \right)^{k-i} (k + 1 - i) x(i) \right\} \\
- 2 \sum_{i=1}^{k-1} \left( \frac{a}{2} \right)^{k-i} (k - i) x(i) + 2 \sum_{i=1}^{k_0} \left( \frac{a}{2} \right)^{k-i} (k - i) x(i) \\
+ \sum_{i=0}^{k-2} \left( \frac{a}{2} \right)^{k-i} (k - 1 - i) x(i) - \sum_{i=0}^{k_0-1} \left( \frac{a}{2} \right)^{k-i} (k - 1 - i) x(i) \right\} \\
= \left\{ x(k) + \sum_{i=2}^{k_0} \left( \frac{a}{2} \right)^{k-i} \left[ (k + 1 - i) - 2(k - i) + (k - 1 - i) \right] x(i) \right\} \\
- \left( \frac{a}{2} \right)^{k-k_0-1} (k - k_0) x(k_0 + 1) - \left( \frac{a}{2} \right)^{k-k_0} (k + 1 - k_0) x(k_0) + 2 \left( \frac{a}{2} \right)^{k-k_0} (k - k_0) x(k_0) \\
- \sum_{i=2}^{k_0} \left( \frac{a}{2} \right)^{k-i} \left[ (k + 1 - i) - 2(k - i) + (k - 1 - i) \right] x(i) \right\} \\
= \left\{ x(k) \right\} - \left\{ \left( \frac{a}{2} \right)^{k-k_0} \left[ (k - 1 - k_0) x(k_0) - \frac{2}{a} (k - k_0) x(k_0 + 1) \right] \right\} \\
= \left\{ x(k) \right\} - s_{k_0} \{ x(k) \}.
\]

\[\square\]

REFERENCES


A discrete non-classical operational calculus model with the Horadam difference


**MODEL DYSKRETNY NIEKLASYCZNEGO RACHUNKU OPERATORÓW Z RÓŻNICĄ HORADAMA**

**STRESZCZENIE**

W artykule skonstruowano model nieklasycznego rachunku operatorów Bittnera, w którym pochodna $S$ związana z ciągami Horadama, rozumiana jest jako operacja różnicowa

$S\{x(k)\} := \{x(k + 2) - a x(k + 1) - b x(k)\}$.

Słowa kluczowe:
rachunek operatorów, pochodna, pierwotne, warunki graniczne, różnica Horadama.