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FREE VIBRATION OF ASYMMETRIC SHAFT

ABSTRACT

The paper deals with free vibrations of a uniform asymmetric shaft. As an example of asymmetry, a shaft of rectangular cross section is chosen. It is assumed that the shaft is simply supported in two bearings and rotates with constant angular velocity, and that the damping forces can be neglected. Bending vibrations of the shaft caused by an initial disturbance of its equilibrium position are considered. For their description the co-rotating coordinate system and the fixed coordinate system are used. The Euler-Bernoulli theory of bending of beams is utilized.

Keywords:
mechanical vibrations, continuous model, free vibrations.

INTRODUCTION

Ship classification societies and designers deal for years with the adverse effects of vibration [3, 7]. Many years of experience proved that vibration can cause:

— damage of the structure or shortening of its durability due to material fatigue;
— incorrect operation or failures in engines and equipment;
— fatigue of the crew and the resulting decrease of the work efficiency;
— noise with its different adverse influence upon the comfort and health of the crew.

The literature also shows that vibration may modify crewmember perception (e.g. reading text and instruments, depth perception), influence task control movements (e.g. tactile sense, head/hand movements, manual tracking) and lead to impairment of speech. These factors may result in increased crewmember reaction/ response times and possibility of human error [2].

Vibration can be classified in several ways. One of the important classifications is that of free and forced vibrations. If a system, after an initial disturbance, is left to vibrate on its own, the ensuing vibration is known as free vibration. Examples of it are the oscillation of a pendulum or the vertical oscillatory motion felt by a bicyclist after hitting a road bump. If a system is subjected to an external time-varying force, the resulting vibration is known as forced vibration. The oscillation that arises in machines such as diesel engines or the vibration produced by an unbalanced rotor are examples of forced vibration.

Important parameters in the design stage or measurement, as well as in evaluation, analysis and prevention of shipboard vibration, are concerned with propeller-induced vibration, machinery-induced vibration and wave-induced vibration [4, 7]. The propeller blades operate in the nonuniform wake field between the ship afterbody and rudder. Consequently, the blade loads vary with time which results in shafting vibration, hull girder vibration and local vibration of the superstructure, mast house etc. as well as in pressure pulses on the stern plating.

The main source of machinery-induced vibration are internal combustion engines (main and auxiliary), where the excitation forces come from [9]:

- pressure pulsations in the inlet and outlet ducts;
- pressure changes in cylinders during the combustion process;
- inertia of moving engine elements;
- operation of the timing gear;
- toothed gears;
- pressure changes in fuel and lubrication systems;
- auxiliary devices.

The usual way to reduce the negative effects of mass forces from moving engine elements is balancing. Counterweights are used primarily to balance single cranks internally. In engines having an even number of cranks they are arranged symmetrically so that the moving masses balance each other out. These engines are said to be externally balanced. In engines with any odd number of cranks special balancing holes in the flywheel, additional weights at the free end of the crankshaft, additional counterclockwise rotating wheel with a balancing mass, non-equidistant angles between different cranks, etc. are used [9]. Also all possible firing orders are analyzed be means of special software and the optimum one is chosen.

The wave-induced load acting on the ship hull will cause primarily the ship to respond in a steady-state motion. In this case, the wave load will induce ship deflection, which includes the rigid body modes as well as her elastic deformation modes. However, heavy seas may cause also free vibration of the ship hull and equipment. In particular, the bottom-impact slamming as a source of high dynamic loads and free vibrations is very important to small and medium-size ships as well as to high-speed vessels.

The subject of this paper are free vibrations of an asymmetric shaft under assumptions that the initial conditions are known and that the damping forces can be neglected. As an example, similarly as in [5] the uniform shaft of rectangular cross section is taken.

DESCRIPTION OF THE VIBRATING SYSTEM

Consider a horizontal shaft of length l and angular velocity ω, simply supported (pinned) in bearings at the ends (fig. 1).

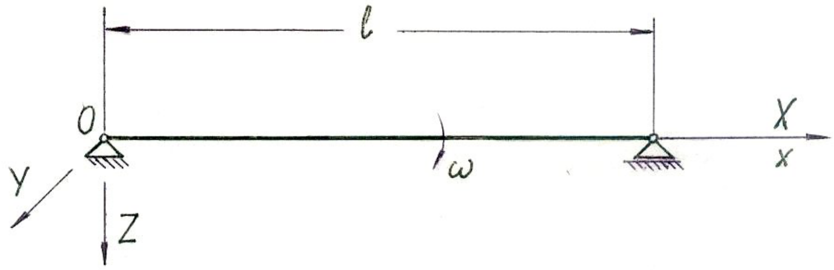

Fig. 1. Scheme of the vibrating system

Source: own study.

For describing its free vibration we shall use the fixed coordinate system *OXYZ* and the co-rotating coordinate system *Oxyz*. The axes X and x are lying along the line of centers of the bearings, Y is the horizontal axis and Z is the vertical axis, and y and z are the axes parallel to the principal axes of inertia of the cross sections of the shaft. In figure 2 the axes of symmetry y' and z' of an arbitrary cross section are shown which are in common with the principal axes of inertia of this cross section. The cross-sectional dimensions are bxh.

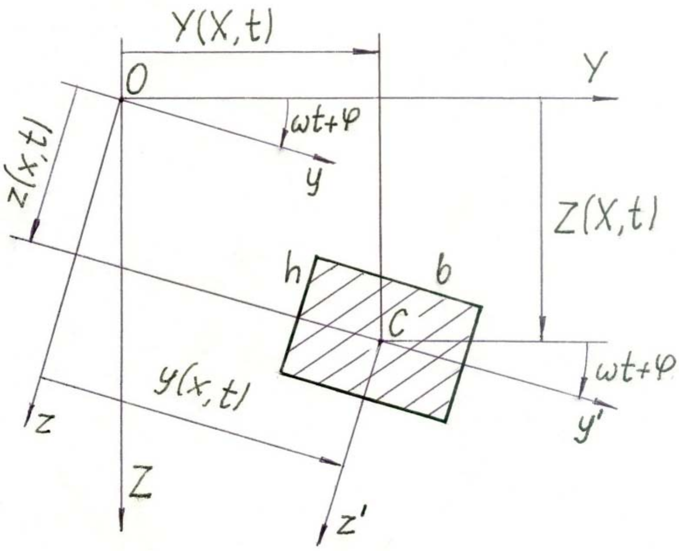

Fig. 2. Translations of the geometric center *C* of an arbitrary cross section of the shaft in displaced condition

Source: own study.

The instantaneous angular position of the shaft is determined by the angle $\omega t + \varphi$ between the axes *y* and *Y*, where φ is thy angle at which the shaft is disturbed from its equilibrium position. The instantaneous deflection of the shaft axis is determined by the coordinates *Y*(*X*, *t*) and *Z*(*X*, *t*) of geometric centers of cross sections of the shaft in the fixed coordinate system or, alternatively, by the coordinates *y*(*x*, *t*) and *z*(*x*, *t*) of these centers in the rotating coordinate system. According to figure 2, the coordinate transformations are:

$$\begin{aligned} Y(X,t) &= Y(x,t) = y(x,t)\cos(\omega t + \varphi) - z(x,t)\sin(\omega t + \varphi) \\ Z(X,t) &= Z(x,t) = y(x,t)\sin(\omega t + \varphi) + z(x,t)\cos(\omega t + \varphi) \end{aligned}. \qquad (1)$$

FREE VIBRATION OF THE CONSIDERED SHAFT

In order to analyse the lateral vibration of the shaft, the Euler-Bernoulli theory of bending of beams [1, 6, 8] can be utilized. In accordance with this theory, the lateral free vibration $w(x,t)$ of a uniform beam is governed by equation:

$$EI\frac{\partial^4 w}{\partial x^4}(x,t)+\rho S\frac{\partial^2 w}{\partial t^2}(x,t)=0\,, \tag{2}$$

where:

E — Young's modulus;
I — axial moment of inertia of the cross section of the beam;
S — cross-sectional area of the beam;
ρ — mass density of the beam.

Since the equation of motion involves a second-order derivative with respect to time and a fourth-order derivative with respect to x, two initial conditions and four boundary conditions are needed for finding a unique solution for $w(x,t)$. Usually, the initial values of lateral displacement and velocity are specified as $w_0(x)$ and $\dot{w}_0(x)$ at $t=0$, so that the initial conditions become

$$w(x,t=0)=w_0(x)\,; \tag{3}$$

$$\frac{\partial w}{\partial t}(x,t=0)=\dot{w}_0(x)\,. \tag{4}$$

The free vibration solution can be found using the method of separation of variables as

$$w(x,t)=W(x)T(t)\,. \tag{5}$$

Substituting Eq. (5) into Eq. (2) and rearranging lead to

$$\frac{EI}{\rho SW(x)}\cdot\frac{d^4W(x)}{dx^4}=-\frac{1}{T(t)}\cdot\frac{d^2T(t)}{dt^2}=\Omega^2\,, \tag{6}$$

where Ω^2 is a positive constant.

Eq. (6) can be written as two equations:

$$\frac{d^4W(x)}{dx^4}-\beta^4W(x)=0\,; \tag{7}$$

$$\frac{d^2T(t)}{dt^2}+\Omega^2T(t)=0\,, \tag{8}$$

where

$$\beta^4=\frac{\rho S\Omega^2}{EI}\,. \tag{9}$$

The solution of Eq. (8) can be expressed as

$$T(t) = K\cos\Omega t + L\sin\Omega t \,, \tag{10}$$

where *K* and *L* are constants that can be found from the initial conditions and Ω is the natural frequency of vibration.

For the solution of Eq. (7) we assume

$$W(x) = Ce^{sx} \,, \tag{11}$$

where *C* and *s* are constants, and derive the auxiliary equation as

$$s^4 - \beta^4 = 0 \,. \tag{12}$$

The roots of this equation are

$$s_{1,2} = \pm\beta \,, \quad s_{3,4} = \pm i\beta \,, \tag{13}$$

where $i = (-1)^{1/2}$.

Hence the solution of Eq. (7) becomes

$$W(x) = C_1\cos\beta x + C_2\sin\beta x + C_3\cosh\beta x + C_4\sinh\beta x \,. \tag{14}$$

The constants $C_1 - C_4$ can be found from the boundary conditions. For a simple supported (pinned) end of the beam the boundary conditions are

$$\text{deflection} = w = 0 \,, \quad \text{bending moment} = EI\,\frac{\partial^2 w}{\partial x^2} = 0 \,. \tag{15}$$

The function $W(x)$ is known as the normal mode function of the beam. For any beam, there will be an infinite number of normal modes $W_n(x)$ with one natural frequency

$$\Omega_n = (\beta_n l)^2 \left(\frac{EI}{\rho S l^4} \right)^{1/2} \tag{16}$$

associated with each normal mode. For a pinned-pinned beam the values of $\beta_n l$ amount to

$$\beta_n l = n\pi \,; \quad n = 1,2,... \tag{17}$$

and the normal modes are given by

$$W_n(x) = \sin\beta_n x = \sin\frac{n\pi x}{l} \,. \tag{18}$$

Consequently, the free vibration solution for a pinned-pinned beam can be written as [1, 6, 8]

$$w(x,t)=\sum_{n=1}^{\infty}\left(K_n\cos\Omega_n t+L_n\sin\Omega_n t\right)\sin\frac{n\pi x}{l}\,, \tag{19}$$

where the constants K_n and L_n can be found from the initial conditions (3) and (4).

If

$$w_0(x)=0\,,\quad \dot{w}_0(x)=v_0\sin\frac{\pi x}{l}\,, \tag{20}$$

then

$$K_n=0\,,\quad L_n=\frac{v_0\sin\frac{\pi x}{l}}{\Omega_n\sin\frac{n\pi x}{l}}\,, \tag{21}$$

so that

$$w(x,t)=v_0\sin\frac{\pi x}{l}\sum_{n=1}^{\infty}\frac{1}{\Omega_n}\sin\Omega_n t\,. \tag{22}$$

Passing now to the free vibration of the considered shaft, for the sake of simplicity it is convenient to project its deflection $w(x,t)$ on the axes y and z of the rotating coordinate system. Then the displacement of the shaft axis is determined by its components $y(x,t)$ and $z(x,t)$ which are governed by equations:

$$\begin{aligned} EI_{z'}\frac{\partial^4 y}{\partial x^4}(x,t)+\rho S\frac{\partial^2 y}{\partial t^2}(x,t)=0 \\ EI_{y'}\frac{\partial^4 z}{\partial x^4}(x,t)+\rho S\frac{\partial^2 z}{\partial t^2}(x,t)=0 \end{aligned}\,, \tag{23}$$

where (see fig. 2)

$$I_{y'}=\frac{bh^3}{12}\quad,\quad I_{z'}=\frac{b^3h}{12}\,. \tag{24}$$

Similarly, the initial conditions (3) and (4) can be resolved into the components $y_0(x)$, $z_0(x)$ and $\dot{y}_0(x)$, $\dot{z}_0(x)$. In the case of Eqs (20) we have (see fig. 3):

$$\begin{aligned} y_0(x)=0\,,\quad \dot{y}_0(x)=v_0\sin\frac{\pi x}{l}\sin\varphi \\ z_0(x)=0\,,\quad \dot{z}_0(x)=v_0\sin\frac{\pi x}{l}\cos\varphi \end{aligned}\,. \tag{25}$$

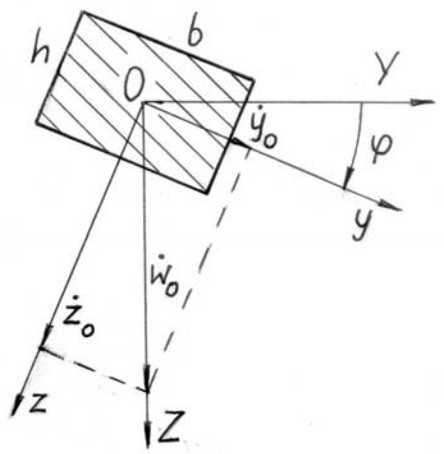

Fig. 3. Components $\dot{y}_0$ and $\dot{z}_0$ of the initial velocity $\dot{w}_0$ of the shaft axis at an arbitrary cross section

Source: own study.

As in Eq. (19), the solutions of Eqs (23) can be determined using the mode superposition principle. For this, the deflections of the shaft are assumed as

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} Y_n(x) y_n(t) \\ z(x,t) &= \sum_{n=1}^{\infty} Z_n(x) z_n(t) \end{aligned}, \qquad (26)$$

where $Y_n(x)$ and $Z_n(x)$ are the *n*-th normal mode functions satisfying equations

$$\begin{aligned} EI_{z'} \frac{d^4 Y_n(x)}{dx^4} - \omega_{yn}^2 \rho S Y_n(x) = 0 \\ EI_{y'} \frac{d^4 Z_n(x)}{dx^4} - \omega_{zn}^2 \rho S Z_n(x) = 0 \end{aligned}. \qquad (27)$$

In Eqs (26), $y_n(t)$ and $z_n(t)$ are the generalized coordinates in the *n*-th modes $Y_n(x)$ and $Z_n(x)$ expressed by

$$\begin{aligned} y_n(t) = K_{yn} \cos \omega_{yn} t + L_{yn} \sin \omega_{yn} t \\ z_n(t) = K_{zn} \cos \omega_{zn} t + L_{zn} \sin \omega_{zn} t \end{aligned} \qquad (28)$$

and

$$\omega_{yn} = (n\pi)^2 \left(\frac{EI_{z'}}{\rho S l^4} \right)^{1/2} , \quad \omega_{zn} = (n\pi)^2 \left(\frac{EI_{y'}}{\rho S l^4} \right)^{1/2} \tag{29}$$

are the n-th natural frequencies of bending vibrations of the shaft in the directions y and z, respectively. That angular velocity of the shaft which equals one of the frequencies (29) is defined in [5] as a critical speed of the shaft.

Following the results obtained for a beam, for a pinned-pinned shaft of rectangular cross section is

$$Y_n(x) = Z_n(x) = \sin \frac{n\pi x}{l}, \tag{30}$$

so that

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} \left(K_{yn} \cos \omega_{yn} t + L_{yn} \sin \omega_{yn} t \right) \sin \frac{n\pi x}{l} \\ z(x,t) &= \sum_{n=1}^{\infty} \left(K_{zn} \cos \omega_{zn} t + L_{zn} \sin \omega_{zn} t \right) \sin \frac{n\pi x}{l} \end{aligned} . \tag{31}$$

In the case of initial conditions (25), the constants in Eqs (31) become

$$\begin{aligned} & K_{yn} = K_{zn} = 0 \\ & L_{yn} = \frac{v_0 \sin \frac{\pi x}{l} \sin \varphi}{\omega_{yn} \sin \frac{n\pi x}{l}} , \quad L_{zn} = \frac{v_0 \sin \frac{\pi x}{l} \cos \varphi}{\omega_{zn} \sin \frac{n\pi x}{l}} . \end{aligned} \tag{32}$$

Consequently, the coordinates of the shaft axis in the rotating coordinate system are

$$\begin{aligned} y(x,t) &= v_0 \sin \frac{\pi x}{l} \sin \varphi \sum_{n=1}^{\infty} \frac{1}{\omega_{yn}} \sin \omega_{yn} t \\ z(x,t) &= v_0 \sin \frac{\pi x}{l} \cos \varphi \sum_{n=1}^{\infty} \frac{1}{\omega_{zn}} \sin \omega_{zn} t \end{aligned} . \tag{33}$$

The solutions (33) can be rewritten as

$$\begin{aligned} y(x,t) &= a(x) b(t) \sin \varphi \\ z(x,t) &= a(x) c(t) \cos \varphi \end{aligned} , \tag{34}$$

where

$$a(x) = v_0 \sin \frac{\pi x}{l}, \quad b(t) = \sum_{n=1}^{\infty} \frac{1}{\omega_{yn}} \sin \omega_{yn} t, \quad c(t) = \sum_{n=1}^{\infty} \frac{1}{\omega_{zn}} \sin \omega_{zn} t . \tag{35}$$

The next step will be the transformation of the solutions (34) to the fixed coordinate system by means of Eqs (1) which yields

$$Y(x,t) = a(x)\left\{\sin\varphi\cos\varphi\left[b(t)-c(t)\right]\cos\omega t - \left[b(t)\sin^2\varphi - c(t)\cos^2\varphi\right]\sin\omega t\right\}; \qquad (36)$$

$$Z(x,t) = a(x)\left\{\left[b(t)\sin^2\varphi + c(t)\cos^2\varphi\right]\cos\omega t + \sin\varphi\cos\varphi\left[b(t)-c(t)\right]\sin\omega t\right\}. \qquad (37)$$

By defining a complex quantity $V(x,t)$ as

$$V(x,t) = Y(x,t) + iZ(x,t) \qquad (38)$$

and by adding Eq. (36) to Eq. (37) multiplied by i , a single equation of lateral motion of the shaft axis is obtained

$$\begin{aligned} V(x,t) = {} & a(x)\left\{\sin\varphi\cos\varphi\left[b(t)-c(t)\right] + ib(t)\sin^2\varphi\right\}(\cos\omega t + i\sin\omega t) + \\ & + ia(x)c(t)\cos^2\varphi(\cos\omega t - i\sin\omega t) \end{aligned} \qquad (39)$$

or, using a vector notation,

$$\bar{V}(x,t) = \bar{u}(x,t)e^{i\omega t} + \bar{v}(x,t)e^{-i\omega t}. \qquad (40)$$

The vectors $\bar{u}(x,t)$ and $\bar{v}(x,t)$ represent the following complex numbers

$$\bar{u}(x,t) = a(x)\left\{\sin\varphi\cos\varphi\left[b(t)-c(t)\right] + ib(t)\sin^2\varphi\right\}; \qquad (41)$$

$$\bar{v}(x,t) = 0 + ia(x)c(t)\cos^2\varphi. \qquad (42)$$

The moduli $u(x,t)$, $v(x,t)$ and arguments $\phi(t)$, $\psi(t)$ of the vectors $\bar{u}(x,t)$ and $\bar{v}(x,t)$ are

$$u(x,t) = \left|a(x)\right|\left\{\left[b(t)-c(t)\right]^2(\sin\varphi\cos\varphi)^2 + b^2(t)\sin^4\varphi\right\}^{1/2}; \qquad (43)$$

$$v(x,t) = \left|a(x)c(t)\right|\cos^2\varphi; \qquad (44)$$

$$\phi(t) = \operatorname{arctg}\frac{b(t)\operatorname{tg}\varphi}{b(t)-c(t)}; \qquad (45)$$

$$\psi(t) = -\frac{\pi}{2}, \quad a(x)c(t) > 0; \quad \psi(t) = \frac{\pi}{2}, \quad a(x)c(t) < 0. \qquad (46)$$

Hence on a complex plane we have the real part of the vector $\overline{V}(x,t)$

$$\mathrm{Re}(\overline{V}) = u(x,t)\cos[\omega t + \phi(t)] + v(x,t)\cos[-\omega t + \psi(t)] \tag{47}$$

and its imaginary part

$$\mathrm{Im}(\overline{V}) = u(x,t)\sin[\omega t + \phi(t)] + v(x,t)\sin[-\omega t + \psi(t)]. \tag{48}$$

So, the lateral motion of the shaft axis at an arbitrary cross section $X = x$ can be described in the fixed coordinate system by the vector

$$\overline{V}(x,t) = \sqrt{\left[\mathrm{Re}(\overline{V})\right]^2 + \left[\mathrm{Im}(\overline{V})\right]^2}\, e^{i\alpha(\overline{V})}, \tag{49}$$

where

$$\alpha(\overline{V}) = \mathrm{arctg}\frac{\mathrm{Im}(\overline{V})}{\mathrm{Re}(\overline{V})}. \tag{50}$$

CONCLUSIONS

The vibratory motion of the considered shaft depends inexplicitly on the parameters of the system and on the initial conditions. In particular, according to Eqs (34), (40), (43) and (44) at $\varphi = 0$ there will be only the free bending vibration $z(x,t)$ of the shaft and the backward whirl of the shaft axis about the line of centers of the bearings of the frequency ω (equal to the angular velocity of the shaft) and time-dependent amplitude

$$v_a(x,t) = |a(x)c(t)|, \tag{51}$$

whereas at $\varphi = \pi/2$ there will be only the free bending vibration $y(x,t)$ of the shaft and the forward whirl of the shaft axis about the line of centers of the bearings of the frequency ω and time-dependent amplitude

$$u_a(x,t) = |a(x)b(t)|. \tag{52}$$

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DRGANIA SWOBODNE ASYMETRYCZNEGO WAŁU

STRESZCZENIE

Artykuł dotyczy swobodnych drgań pryzmatycznego asymetrycznego wału. Jako przykład asymetrii przyjęto wał o przekroju prostokątnym. Założono, że wał jest podparty przegubowo w dwóch łożyskach i wiruje ze stałą prędkością kątową oraz że tłumienie w układzie może być pominięte. Rozpatrywane są giętne drgania wału wywołane początkowym zakłóceniem jego położenia równowagi. Do ich opisu przyjęto układ współrzędnych wirujący wraz z wałem oraz nieruchomy układ współrzędnych. Zastosowano teorię zginania belek Eulera-Bernoulliego.

Słowa kluczowe:

drgania mechaniczne, model ciągły, drgania swobodne.